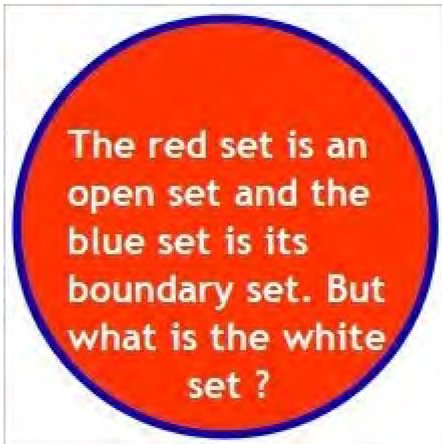

You cannot paint a picture without a canvas

Smooth Manifolds

Definition 1.1.1 (Smooth m-Manifold). *Let $m \in \mathbb{N}_0$. A* **smooth m-manifold** *is a topological space M, equipped with an open cover $\{U_\alpha\}_{\alpha \in A}$ and a collection of homeomorphisms $\phi_\alpha : U_\alpha \to \Omega_\alpha$ onto open sets $\Omega_\alpha \subset \mathbb{R}^m$ (see Figure 1.1) such that, for each pair $\alpha, \beta \in A$, the transition map*

$$\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{1.1.2}$$

is smooth. The homeomorphisms ϕ_α are called **coordinate charts** *and the collection $\mathscr{A} := \{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ is called an* **atlas***.*

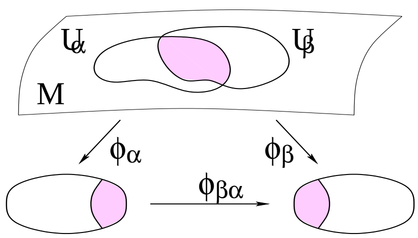

Figure 1.1: Coordinate charts and transition maps.

Let $(M, \mathscr{A} = \{U_\alpha, \phi_\alpha\}_{\alpha \in A})$ be a smooth m-manifold. Then a subset $U \subset M$ is open if and only if $\phi_\alpha(U \cap U_\alpha)$ is an open subset of $\mathbb{R}^m$ for every $\alpha \in A$. Thus the topology on M is uniquely determined by the atlas. A homeomorphism $\phi : U \to \Omega$ from an open set $U \subset M$ to an open set $\Omega \subset \mathbb{R}^m$ is called **compatible with the atlas** $\mathscr{A}$ if the transition map $\phi_\alpha \circ \phi^{-1} : \phi(U \cap U_\alpha) \to \phi_\alpha(U \cap U_\alpha)$ is a diffeomorphism for each α. The atlas $\mathscr{A}$ is called **maximal** if it contains every coordinate chart that is compatible with all its members. Thus every atlas $\mathscr{A}$ is contained in a unique maximal atlas $\overline{\mathscr{A}}$, consisting of all coordinate charts $\phi : U \to \Omega$ that are compatible with $\mathscr{A}$. Such a maximal atlas is also called a **smooth structure** on the topological space M. We do not distinguish the manifolds $(M, \mathscr{A})$ and $(M, \mathscr{A}')$ if the corresponding maximal atlasses agree, i.e. if the charts of $\mathscr{A}'$ are all compatible with $\mathscr{A}$ (and vice versa) or, equivalently, if the union $\mathscr{A} \cup \mathscr{A}'$ is again a smooth atlas. If this holds, we say that $\mathscr{A}$ and $\mathscr{A}'$ induce the same smooth structure on M.

Differential Geometry

Differentiable Manifolds

- Definition of **TOPOLOGICAL MANIFOLD** : It is a topological space (E, τ) so that

  1. It is Hausdorff.
  2. $\forall x \in E$ there exists (U, φ) with U open and $x \in U$, such that $\varphi : U \to \varphi(U)$ is a homeomorphism. The pair (U, φ) is called **CHART** and the real numbers $(x^1, \ldots, x^n) = \varphi(x)$ are called **LOCAL COORDINATES**.
  3. (E, τ) has a countable basis of open sets.

- Remark. Topological Manifolds are paracompact, i.e, every open cover has a locally finite refinement.

- Remark. Paracompactness implies having Partition of unity.

- Definition. A **DIFFERENTIABLE** $[C^\infty, C^k, C^\omega]$ **STRUCTURE** on a topological manifold M is a family of charts $\mathcal{U} = \{(U_\alpha, \varphi_\alpha\}$ so that

  1. $\bigcup U_\alpha = M$
  2. If $U_\alpha \cap U_\beta \neq \emptyset$ then $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are differentiable $[C^\infty, C^k, C^\omega]$ with differentiable $[C^\infty, C^k, C^\omega]$ inverse. In this case we say that $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are **COMPATIBLE**.
  3. Completness property: If (V, ψ) is a chart which is compatible with every $(U_\alpha, \varphi_\alpha) \in \mathcal{U}$ then $(V, \psi) \in \mathcal{U}$.

- Remark.It is not necessary to verify the third property because of the following proposition.

- **Proposition.** Let M be Hausdorff with countable basis of open sets. Let $\{(V_\alpha, \psi_\alpha) : \alpha \in A\}$ be a covering of M by C^∞-compatible coordinate charts. Then $\exists!$ C^∞ structure containing the charts $\{(V_\alpha, \psi_\alpha) : \alpha \in A\}$.

- Examples of topological manifolds:

  1. $(\mathbb{R}^n, Id)$
  2. If M is a C^∞ manifold and $N \subset M$ is an open subset then N is a $\mathcal{C}^\infty$ manifold too.